Lecture 8 on Oct. 3

We have learned that a linear transformation maps circles to circles. There is one more question left unsolved. If a linear transformation, denoted by S is given and a circle, denoted by C is given, how can we determine the image of this circle C under the action of the linear transformation S. To determine a circle, we only need to know its center and the radius. Given a point z_0 on C, clearly $S(z_0)$ lies on the image of C. Therefore if we know where the center of the image of C is, then the radius of the imaging circle can be found by the absolute value of $S(z_0) - p_0$. Here p_0 is the center of the imaging circle. All the above arguments reduce the problem to search the center of the imaging circle.

Before proceeding, let us introduce the concept of symmetric point.

Definition 1. *Given z_2, z_3, z_4, we can determine a line or a circle, denoted by C, passing these three points. If z is arbitrarily given, then z^* is called the symmetric point of z with respect to the circle C if z^* satisfies the following equation*

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}.$$

One should notice that the above definition is independent of the choice of z_2, z_3 and z_4. That is

Remark 1. *if (z_2', z_3', z_4') and (z_2, z_3, z_4) determine an identical circle C, then the two symmetric points given by Definition 1 are equal.*

Proof. the proof is just a straightforward calculation. here we only assume $z_2' \neq z_2$ and let $z_3' = z_3$, $z_4' = z_4$. If $(z^*)'$ is the symmetric point of z given by the triple (z_2', z_3, z_4), then by Definition 1, we have

$$\frac{z_2' - z_3}{z_2' - z_4} \Big/ \frac{(z^*)' - z_3}{(z^*)' - z_4} = \overline{\left(\frac{z_2' - z_3}{z_2' - z_4} \Big/ \frac{z - z_3}{z - z_4} \right)} \tag{0.1}$$

Since z_2' stays on C, then we know that $(z_2', z_2, z_3, z_4) = \lambda$ is real. Equivalently we have

$$\frac{z_2' - z_3}{z_2' - z_4} = \lambda \frac{z_2 - z_3}{z_2 - z_4}.$$

Applying the above equality to (0.1) and noticing that λ is real, one can easily show that

$$\frac{z_2 - z_3}{z_2 - z_4} \Big/ \frac{(z^*)' - z_3}{(z^*)' - z_4} = \overline{\left(\frac{z_2 - z_3}{z_2 - z_4} \Big/ \frac{z - z_3}{z - z_4} \right)} \tag{0.2}$$

Still by Definition 1, $(z^*)'$ is the symmetric point of z given by the triple (z_2, z_3, z_4). The proof is done. □

Since cross ratio is invariant under linear transformations, it holds

$$(Tz^*, Tz_2, Tz_3, Tz_4) = \overline{(Tz, Tz_2, Tz_3, Tz_4)}.$$

By the Definition 1, Tz^* must be the symmetric point of Tz. Therefore, we conclude that

Proposition 1. *linear transformation maps symmetric pair to symmetric pair. More precisely if (z, z^*) is a symmetric pair with respect to the circle determined by z_2, z_3 and z_4, then (Tz, Tz^*) is a symmetric pair with respect to the circle determined by Tz_2, Tz_3 and Tz_4.*

In fact, the concept of symmetric point is not new to us. In the following arguments, we still use the notations in Definition 1. If we assume C is a straight line, then we know that ∞ must be on C. Therefore we can assume $z_3 = \infty$. by Definition 1, we know that

$$\frac{z^* - z_4}{z_2 - z_4} = \overline{\left(\frac{z - z_4}{z_2 - z_4}\right)}. \tag{0.3}$$

If $z_4 = 0$ and $z_2 = 1$, then C is just the x-axis. From (0.1), we see that $z^* = \overline{z}$. They are symmetric with respect to the x-axis. For arbitrary z_2 and z_4, we can also show that z and z^* are symmetric with respect to the line given by z_2 and z_4. In fact if we take absolute values on both sides of (0.1), we get $|z - z_4| = |z^* - z_4|$. By Remark 1, z_4 can be arbitrary point on C, therefore z^* can only be z or the symmetric point of z with respect to C. If $z = z^*$, then by (0.1) we know that $\text{Im}((z - z_4)/(z_2 - z_4)) = 0$. this shows that z is located on the line C. In other words, if z is not on C, z^* must be different from z. That is z^* must be the symmetric point of z with respect to C.

If C is a circle with center a and radius R, then we have

$$\overline{(z, z_2, z_3, z_4)} = \overline{(z - a, z_2 - a, z_3 - a, z_4 - a)} = (\bar{z} - \bar{a}, \bar{z}_2 - \bar{a}, \bar{z}_3 - \bar{a}, \bar{z}_4 - \bar{a}). \tag{0.4}$$

Here the first equality comes from Proposition 0.9 in lecture note 7. Noticing that z_2, z_3 and z_4 are located on C, therefore, we have

$$|z_j - a|^2 = R^2, \qquad j = 2, 3, 4.$$

Applying the above equalities to (0.4), we deduce that

$$\begin{aligned}\overline{(z, z_2, z_3, z_4)} &= (\bar{z} - \bar{a}, \frac{R^2}{z_2 - a}, \frac{R^2}{z_3 - a}, \frac{R^2}{z_4 - a}) = (\frac{R^2}{\bar{z} - \bar{a}}, z_2 - a, z_3 - a, z_4 - a) \\ &= (\frac{R^2}{\bar{z} - \bar{a}} + a, z_2, z_3, z_4).\end{aligned}$$

The above arguments show that

Proposition 2. *If z^* is the symmetric point of z with respect to the circle centering a and having radius R, then*

$$z^* = a + \frac{R^2}{\bar{z} - \bar{a}}$$

finally, let us take a look at some examples.

Example 1: The symmetric point of a point on the circle C with respect to C is itself.

Example 2: If C is a circle centering at a, then the symmetric point of a with respect to C is ∞.

Example 3: Given a circle C, the map from z to z^* is called reflection. Reflect the imaginary line with respect to the circle $|z-2| = 1$.

Solution: Let w be a point on the reflection. then its symmetric point with respect to $|z-2| = 1$ must be on the imaginary line. By Proposition 2, we know that

$$2 + \frac{1}{\bar{w} - 2}$$

must be pure imaginary. Assume $w = w_1 + iw_2$, it is clear from the above equality that

$$2(w_1 - 2)^2 + 2w_2^2 + w_1 - 2 = 0.$$

It is a circle centering at $(7/4, 0)$ with radius $1/4$.

Example 4: Given the unit circle $|z| = 1$ and a linear transformation $Sz = z/(z+2)$. Find out the image of the unit circle under the given linear transformation.

Solution: Pick up one point on the unit circle, say 1. Its image under the action of the linear transformation is $1/3$. By Example 2, $(0, \infty)$ is a symmetric pair with respect to the unit circle. Then by Proposition 1, $(0, 1)$ is symmetric with respect to the image circle. By Proposition 2, if a is the center of the imaging circle. R is its radius. Then we have

$$a_1 - a_1^2 + R^2 = 0, \qquad a_2 = 0. \tag{0.5}$$

Here we assume $a = a_1 + ia_2$. Moreover $1/3$ is on the image circle, therefore $|a - 1/3|^2 = R^2$. Connecting this equation with the (0.5), we get $a_1 = -1/3$ and $R = 2/3$.

Example 5: Find linear transformation which carries $|z| = 2$ to $|z+1| = 1$, the point -2 to the origin, the origin to i.

Solution: Since $(0, \infty)$ is a symmetric pair of $|z| = 2$, it holds that $(i, T\infty)$ is symmetric pair of $|z+1| = 1$. Here T is the linear transformation we are searching. By Proposition 2, we can easily show that $T\infty = (-1+i)/2$. Since $-2 \longmapsto 0$, $0 \longmapsto i$, $\infty \longmapsto (-1+i)/2$, T can be explicitly written out as follows: $Tz = (z+2)/((-1-i)z - 2i)$.